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General Scheme of Measurement Processes

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Quantum mechanical operations are motivated and their formal representation is derived from principles of statistics as well as from interaction processes.

1. INTRODUCTION

For almost twenty years the problem of quantum measurement did not attract the interest of a broader physical community. The development of quantum optics, especially the progress with optical waveguides, which opened possibilities for optical communication systems, gave a new impetus to work in this field. Now quantum measurement theory is fundamental to quantum communication and quantum information theory. In the following I will confine myself to describing operations. Operations are the simplest nontrivial quantum measurements. They show the main features and difficulties of the theory of quantum measurements. Operations can be the starting point of more detailed investigations. The main material presented here is taken from my dissertation (Hellwig, 1967, 1969, 1971), some papers together with Kraus (Hellwig and Kraus, 1969, 1970, 1971), later papers by Kraus (1971, 1977), who discovered the complete positivity of the representing maps, as well as his book Kraus (1983), in which this knowledge is collected. Further recommended literature is Pauli (1933), Davies and Lewis (1970), Ludwig (1976), Davies (1976), Gudder (1979), and Busch *et al.* (1991).

2. SOME GENERALITIES

Usually, the result of a measurement is understood as a statement about the presence or absence of some accidental property at an individual physical

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system on which the measurement has been performed. This is well illustrated in classical mechanics. Here the essence of the physical system together with the external conditions is encoded in a symplectic manifold $\mathcal{M}$ and a Hamiltonian function. The accidence of an individual system is represented by a point of $\mathcal{M}$ which is unknown in general. The result of a measurement consists in the statement that this point is contained in a subset $\alpha \subseteq \mathcal{M}$ which may be the preimage of some function on $\mathcal{M}$, the value of which has been measured.

The philosophy of classical individual systems states that any such system at any time is situated at a point $p \in \mathcal{M}$, i.e., exactly the accidentals α containing p are present and any other is absent. For any accidental it is decided whether it is present or absent. On the other hand, it is impossible to prepare the system in such a way that p results with certainty. The result of a real preparation can at most be a probability distribution described by a probability measure on the Borel sets of $\mathcal{M}$ which is absolutely continuous with respect to the Lebesgue measure. Let us consider a preparation procedure by which the distribution on $\mathcal{M}$ is given by the probability measure

$$\mu: \mathfrak{B}(\mathcal{M}) \to [0, 1]$$

where $\mathfrak{B}(\mathcal{M})$ denotes the Borel sets of $\mathcal{M}$.

A measurement answering the question

$$\begin{array}{ccc} m \in \alpha & \text{or} & m \notin \alpha \\ \text{'yes'} & \text{or} & \text{'no'} \end{array}$$

where $\alpha \in \mathfrak{B}(\mathcal{M})$ and $\mu(\alpha) \in [0, 1]$, decomposes the ensemble into the statistical mixture of two subensembles according to the classical formula of Bayes:

$$\mu = \mu(\alpha) \frac{\mu(\cdot \cap \alpha)}{\mu(\alpha)} + \mu(\mathcal{M}\backslash\alpha) \frac{\mu(\cdot \cap (\mathcal{M}\backslash\alpha))}{\mu(\mathcal{M}\backslash\alpha)}$$

The subensembles with the probability measures

$$\hat{\mu}_+ := \frac{\mu(\cdot \cap \alpha)}{\mu(\alpha)}, \qquad \hat{\mu}_- := \frac{\mu(\cdot \cap (\mathcal{M}\backslash\alpha))}{\mu(\mathcal{M}\backslash\alpha)}$$

can be prepared by selecting those individual systems for which the result of the measurement is 'yes' or 'no,' respectively. Bayes' formula expresses the nondisturbance principle of classical measurements in that the set of accidentals present at an individual system is the same before and after the measurement.

The classical philosophy about the accidence of physical systems does not apply to quantum physics. Nevertheless, selection procedures with respect

to 'yes'–'no' experiments like the classical one just described can still be performed. They are called quantum mechanical operations. However, the 'nondisturbance principle' and the formula of Bayes do not hold generally. The description of operations leads to generalizations of the Bayes formula which include the necessary quantal state changes.

Instead of the symplectic manifold $\mathcal{M}$ with a distinguished Hamiltonian function we have, as the result of a quantization procedure, a complex Hilbert space $\mathcal{H}$ with a distinguished Hamiltonian operator H. The Boolean algebra formed by the characteristic functions of the Borel sets of $\mathcal{M}$ is in a sense replaced by the lattice formed by the orthogonal projections $\mathcal{P}(\mathcal{H})$, the so-called quantum logic. Recalling that the self-adjoint operators correspond to their spectral measures, quantum logics at first sight seem to extend classical Boolean logics to a more general one, but from the physical point of view this is a restriction because of the incompatibility of position and momentum. What is the accidence of an individual quantum system?

To change the classical philosophy into a suitable one for quantum systems hidden variables have been invented. As shown by the experiments of A. Aspect improving the quantal form of Bell's inequality in Bohm's form of the Einstein–Podolski–Rosen situation, hidden variables cannot be maintained together with the principle of locality. Hence, believing in the absence of action at a distance, one has to forget about hidden variables. But also smaller sets of 'elements of physical reality' in the sense of Einstein, Podolski, and Rosen are ruled out by such experiments.

To understand the following we need not enter into the deep philosophical problems about quantum reality. We only need to speak about:

- Preparation procedures leading to probability measures on $\mathcal{P}(\mathcal{H})$.
- Registration of classically observable effects on macroscopic apparatuses occurring after interaction with a quantum object.
- Selection procedures which render subensembles according to occurred or not occurred effects.

The task is to characterize the generalizations of the classical procedure and of Bayes' formula without using the classical philosophy about the realization of the accidence and the nondisturbance principle.

3. OPERATIONS AND EFFECTS

The fundamental concepts of a statistical theory are:

- A convex structure $\tilde{\mathfrak{S}}$ which represents the set of preparation procedures involved in the theory.

- A set $\tilde{\mathfrak{E}}$ representing the set of 'yes'–'no' registration procedures including a trivial registration procedure which always answers 'yes.'
- A probability law $\tilde{\mu}$: $\tilde{\mathfrak{S}} \times \tilde{\mathfrak{E}} \to [0, 1]$.

A trivial 'yes–no' registration procedure $\tilde{e}$ fulfils $\tilde{\mu}(\cdot, \tilde{e}) = 1$. One forms classes of equivalent preparation procedures and classes of equivalent 'yes'–'no' registration procedures in the obvious way and introduces:

- The convex structure of states $\mathfrak{S}$ consisting of the classes of equivalent preparation procedures.
- The set of effects $\mathfrak{E}$ consisting of the classes of equivalent registration procedures; the class of trivial registration procedures shall be denoted by e.
- The probability law μ: $\mathfrak{S} \times \mathfrak{E} \to [0, 1]$.

For the trivial effect e, $\mu(\cdot, e) = 1$. The advantage of this factorization is that the set of effects is separating the set of states and the set of states is separating the set of effects. By a well-known and simple construction one shows that $\mathfrak{S}$, $\mathfrak{E}$, μ, and e can be identified with objects in a dual pair of partially ordered normed complete vector spaces $\mathfrak{B}$, $\mathfrak{B}'$, where:

- $\mathfrak{S}$ forms a base of the convex cone $\mathfrak{C}^+$ of positive elements of $\mathfrak{B}$, the norm of which is the base norm.
- e is the order unit of $\mathfrak{B}'$, $\mathfrak{E}$ is its order unit segment, and the norm is the order unit norm.
- μ is just the restriction to $\mathfrak{S} \times \mathfrak{E}$ of the bilinear pairing $\langle \cdot, \cdot \rangle$ of $(\mathfrak{B}, \mathfrak{B}')$.

Although the following can be formulated in this abstract setting, we will confine ourselves to the Hilbert space model of quantum mechanics. Let $\mathcal{H}$ denote a Hilbert space; then:

- $\mathfrak{B}$ is the space $\mathfrak{B}_s(\mathcal{H})_\infty$ of Hermitian trace class operators in $\mathcal{H}$ with the trace norm $\|T\| := \mathrm{tr}(|T|)$, $T \in \mathfrak{B}_s(\mathcal{H})_\infty$.
- $\mathfrak{B}'$ is the space of bounded Hermitian operators $\mathfrak{B}_s(\mathcal{H})$ in $\mathcal{H}$ with operator norm.
- $\langle T, A \rangle := \mathrm{tr}(T, A)$, $T \in \mathfrak{B}_s(\mathcal{H})_\infty$, $A \in \mathfrak{B}_s(\mathcal{H})$.
- $\mathfrak{S}$ is the set of positive elements $W \in \mathfrak{B}_s(\mathcal{H})_\infty$ with $\mathrm{tr}(W) = 1$.
- $\mathfrak{E} := \{F \in \mathfrak{B}_s(\mathcal{H}) | 0 \leq F \leq 1\}$.
- e is the unit operator 1 of $\mathcal{H}$.

In the following let $W \in \mathfrak{S}$ and $F, G \in \mathfrak{E}$.

It is useful to have a formal scheme of a 'yes'–'no' measurement in mind (Fig. 1), showing preparation and registration apparatuses as black boxes. Let N_+ be the number of results 'yes' and N_- be the number of results 'no' in a series of N experiments. Then for $N \to \infty$

Fig. 1.

$$\frac{N_+}{N} \to \mathrm{tr}(WF), \qquad \frac{N_-}{N} \to \mathrm{tr}(W(1 - F))$$

A nonselective operation is a 'yes'–'no' experiment which does not absorb the objects such that subsequent experiments with them can be performed. The formal scheme of a nonselective operation is shown in Fig. 2, where $W \to \hat{W}$ describes the state change caused by the operation and G is the operator of a subsequent measurement.

In a selective operation the objects which cause the answer 'no' will be absorbed while the objects causing the answer 'yes' become free thereafter and are available for subsequent experiments. By this selection procedure the state change $W \to \hat{W}_+$ is caused. The formal scheme is shown in Fig. 3.

Analogously, a selective operation can be considered in which the objects causing '$-$' become free thereafter and those causing '$+$' are absorbed. The state prepared by this procedure is denoted by $\hat{W}_-$.

If the classical nondisturbance principle held, the density operators W, $\hat{W}_+$, $\hat{W}_-$, and F would be related by the formula of Bayes. Since this principle fails to hold, we have to look for more general relations.

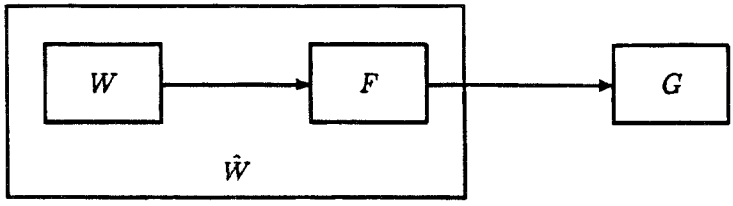

Fig. 2.

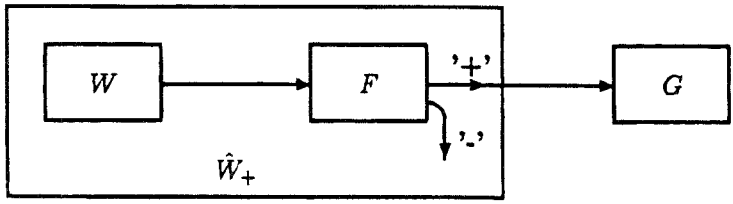

Fig. 3.

Obviously we can state

$$\hat{W} = \begin{cases} \hat{W}_+, & \operatorname{tr}(WF) = 1 \\ \hat{W}_-, & \operatorname{tr}(WF) = 0 \\ \operatorname{tr}(WF)\hat{W}_+ + \operatorname{tr}(W(1-F))\hat{W}_-, & \text{otherwise} \end{cases}$$

J. von Neumann and G. Lüders (Lüders, 1951) supposed in the case that the effect operator is a projection operator E that

$$\hat{W}_+ = \frac{EWE}{\operatorname{tr}(WF)}, \qquad \hat{W}_- = \frac{(1-E)W(1-E)}{\operatorname{tr}(W(1-F))}$$

This assumption is often called the "minimal disturbance principle." These formulas presuppose that the effect is presented by a projection operator and do not make sense for general effect operators F, $0 \leq F \leq 1$.

Instead of the projection postulate

$$W \mapsto EWE$$

we now introduce a mapping

$$\Phi: \mathfrak{S} \to \mathfrak{C}^+ \cup \{0\} \subseteq \mathfrak{B}$$

$$W \mapsto \begin{cases} \operatorname{tr}(WF)\hat{W}_+, & \operatorname{tr}(WF) \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

which makes sense also in the general case. Observe now that the projection postulate is the restriction to $\mathfrak{S}$ of a complex linear map of the complex space $\mathfrak{B}(\mathcal{H})_1$ into itself, which is, moreover, completely positive. We will show by simple assumptions that both properties also hold true for the mapping Φ and that the set of mappings characterized by these two properties appear as the natural generalizations of the von Neumann–Lüders postulates. Furthermore, this set of mappings just comes out when the measurement is understood as a result of an interaction process.

By the very definition of statistical mixtures it is clear that the mapping Φ must be affine, i.e., for $S_1, S_2 \in \mathfrak{S}$ and $0 \leq \lambda \leq 1$ there holds

$$\Phi(\lambda S_1 + (1-\lambda)S_2) = \lambda\Phi(S_1) + (1-\lambda)\Phi(S_2)$$

One checks easily that

$$\Phi_+: \mathfrak{C}^+ \cup \{0\} \to \mathfrak{C}^+ \cup \{0\}$$

$$T \mapsto \begin{cases} 0, & T = 0 \\ (\operatorname{tr} T)\Phi\left(\dfrac{T}{\operatorname{tr} T}\right), & \text{otherwise} \end{cases}$$

extends Φ from $\mathfrak{S}$ to $\mathfrak{C}^+ \cup \{0\} \subseteq \mathfrak{B}_s(\mathcal{H})_1$, where $\mathfrak{C}^+$ is the positive cone of the real space of the symmetric trace class operators, and this extension is homogeneous for positive numbers and additive. Since the positive cone of $\mathfrak{B}_1(\mathcal{H})$ is generating, i.e., any $T \in \mathfrak{B}_1(\mathcal{H})$ can be written as $T = T_+ - T_-$, $T_+, T_- \in \mathfrak{C}^+ \cup \{0\}$, it is almost trivial to check that

$$\Phi_r(T) := \Phi_r(T_+) + \Phi_r(T_-)$$

is well defined and a linear extension of Φ_+ from $\mathfrak{C}^+ \cup \{0\}$ to $\mathfrak{B}_s(\mathcal{H})_1$, which is positive by construction. By

$$\Phi_c(T_1 + iT_2) := \Phi_r(T_1) + i\Phi_r(T_2)$$

this map extends linearly to the complex space of all trace class operators. Finally, one can show that Φ_c is bounded with respect to the trace norm, i.e.,

$$\|\Phi(T)\|_1 \leq C\|T\|_1, \qquad \|T\|_1 = \mathrm{tr}[(T + T)^{1/2}]$$

and that a bound is given by

$$C = \sup_{W \in \mathfrak{S}} \mathrm{tr}(\Phi(W)) \leq 1$$

By construction there holds for $F \in \mathfrak{E}$ and $W \in \mathfrak{S}$

$$\mathrm{tr}(WF) = \mathrm{tr}(\Phi(W))$$

Since the space of bounded linear operators $\mathfrak{B}(\mathcal{H})$ is just the Banach dual of the space $\mathfrak{B}(\mathcal{H})_1$ with the trace norm and the extension Φ_c of Φ is bounded, we can introduce uniquely the dual mapping

$$\Phi_c^*: \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})$$

by the requirement that for any $T \in \mathfrak{B}(\mathcal{H})_1$

$$\mathrm{tr}(T\Phi_c^*(X)) = \mathrm{tr}(\Phi_c(T)X), \qquad X \in \mathfrak{B}(\mathcal{H})$$

Φ_c^* is a linear, operator-norm bounded, positive map with

$$\|\Phi_c^*(X)\| \leq (\sup_{W \in \mathfrak{S}} \mathrm{tr}(\Phi(W))\|X\|$$

such that the bound is less than one. Obviously there holds

$$F = \Phi_c^*(1)$$

Summing up to this point, we have established, by the purely statistical argument that convex mixtures of preparations can always be prepared, a relation between $W \in \mathfrak{S}$, $\hat{W}_+$, and F, namely, that there is a linear, bounded, positive operator

$$\Phi_c: \mathfrak{B}(\mathcal{H})_1 \to \mathfrak{B}(\mathcal{H})_1$$

such that

$$\hat{W}_+ = \frac{\Phi_c(W)}{\mathrm{tr}(FW)}$$

Remember that $\mathrm{tr}(FW) = \mathrm{tr}(\Phi_c(W))$, and $F = \Phi_c^*(1)$.

This relation becomes much more specific after establishing the complete positivity of the map Φ_c. If one considers the state change as the result of an interaction with another quantum system, the apparatus on which an observation is described by the von Neumann–Lüders assumption, this property is a formal consequence. But one may also establish it more directly, as we will do now.

Any two quantum systems can be considered as being coupled one to the other such that they are described in the common Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$. The reason for the tensor product structure is the fact that the observables of each component are observables of the composed system and two observables concerning different subsystems have to be commensurable. Consider now our quantum object described in the Hilbert space $\mathcal{H}$ to be coupled with an n-level system described in $\mathbf{C}^n$. By the argument which has been used to establish the tensor product structure $\mathcal{H} \otimes \mathbf{C}^n$ one may assume that operations concerning only the object described in $\mathcal{H}$ must be extended to the composed system in such a manner that at least when correlations are absent the state of the n-level component is not affected.

To that end we realize that the density operators in $\mathcal{H} \otimes \mathbf{C}^n$ are limit points of linear combinations of uncorrelated density operators like $W \otimes V$, $W \in \mathfrak{S}(\mathcal{H})$, $V \in \mathfrak{S}(\mathbf{C}^n)$ and have the shape

$$\begin{pmatrix} W_{11} & W_{12} & \cdots & W_{1n} \\ W_{21} & W_{22} & \cdots & W_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ W_{n1} & W_{n2} & \cdots & W_{nn} \end{pmatrix}, \qquad W_{ij} \in \mathfrak{B}(\mathcal{H})_1$$

where

$$\mathrm{tr}((W_{ij})) = \sum_{i=1}^{n} \mathrm{tr}\, W_{ii} = 1$$

Especially, an uncorrelated state has the shape

$$W \otimes ((v_{ij})) = \begin{pmatrix} Wv_{11} & Wv_{12} & \cdots & Wv_{1n} \\ Wv_{21} & Wv_{22} & \cdots & Wv_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ Wv_{n1} & Wv_{n2} & \cdots & Wv_{nn} \end{pmatrix}$$

Hence, an operation Φ on the object Hilbert space $\mathcal{H}$ must obviously be extended to act on uncorrelated states in $\mathcal{H} \otimes \mathbf{C}^n$ in such a way that

$$\Phi_n\colon W \otimes ((v_{ij})) \mapsto \Phi(W) \otimes ((v_{ij}))$$
$$= \begin{pmatrix} \Phi(W)v_{11} & \Phi(W)v_{12} & \cdots & \Phi(W)v_{1n} \\ \Phi(W)v_{21} & \Phi(W)v_{22} & \cdots & \Phi(W)v_{2n} \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots \\ \Phi(W)v_{n1} & \Phi(W)v_{n2} & \cdots & \Phi(W)v_{nn} \end{pmatrix}$$

Since Φ is linear and bounded, the mappings just defined for uncorrelated states can be extended to a linear and bounded map onto the linear hull of the uncorrelated density operators, and because it is bounded, it can be continuously extended onto the space of density operators in $\mathcal{H} \otimes \mathbf{C}^n$ to give

$$\Phi_n\colon \begin{pmatrix} W_{11} & W_{12} \cdots & W_{1n} \\ W_{21} & W_{22} \cdots & W_{2n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ W_{n1} & W_{n2} \cdots & W_{nn} \end{pmatrix} \mapsto \begin{pmatrix} \Phi(W_{11}) & \Phi(W_{12}) & \cdots \Phi(W_{2n}) \\ \Phi(W_{21}) & \Phi(W_{22}) & \cdots \Phi(W_{1n}) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \Phi(W_{n1}) & \Phi(W_{n2}) & \cdots \Phi(W_{nn}) \end{pmatrix}$$

Now the representation of an operation Φ_n has to be a positive map, hence, the operation Φ on the object acting on $\mathcal{H}$ is by definition an n-positive map. Since there is no restriction to the number n of levels, Φ has to be completely positive.

As a consequence of the Stinespring representation theorem for completely positive mappings there is a series $\{A_k\}_{k \in K \subseteq \mathbf{N}}$ of linear operators in $\mathcal{H}$ fulfilling

$$\sum_{k \in K} A_k^+ A_k \leq 1$$

such that

$$\Phi(W) = \sum_{k \in K} A_k W A_k^+$$

In Table I the main formulas for the von Neumann–Lüders operation are compared with those of a general operation.

Up to now, no quantum dynamic principles have been taken into consideration. The complete positivity has been established using only general statistical and quantum kinematic principles.

4. EXPLICIT EXPRESSIONS OF THE A_k FOR A MEASURING PROCESS

The measuring process will be treated as an interaction process between the object and the measuring apparatus. Both systems are quantum systems in principle. That the measuring apparatus has an additional structure as a

Table I.

Operation	von Neumann–Lüders	General
Mathematical representative	$E = E^+ = E^2 \leq 1$	$\{A_k\}_{k\in K\subseteq \mathbb{N}}$, $\Sigma_{k\in K} A_k^+ A_k \leq 1$
Probability for the result '+'	$\mathrm{tr}(WE)$, $E \leq 1$	$\mathrm{tr}(WF)$, $F := \Sigma_{k\in K} A_k^+ A_k \leq 1$
Selection according to the result '+'	$W \mapsto \Phi(W) = EWE$	$W \mapsto \Phi(W) = \Sigma_{k\in K} A_k W A_k^+$
Operator of two subsequent measurements	$G \mapsto \Phi^*(G) = EGE$	$G \mapsto \Phi^*(G) = \Sigma_{k\in K} A_k^+ G A_k$
No subsequent measurement	$E = \Phi^*(1) = E^+E = E$	$F = \Phi^*(1) = \Sigma_{k\in K} A_k^+ A_k$

many-particle system with a macroscopically observable decomposition of the unit operator does not enter into the following computation. Denote by:

- $\mathcal{H}_o$ the Hilbert space of the quantum objects.
- $\mathcal{H}_a$ the Hilbert space of the apparatus.
- $\mathcal{B}(\mathcal{H}_o) \supseteq \mathfrak{S}_o \ni W_o$ the density operator of the ensemble of objects on which the measurements are performed.
- $\mathcal{B}(\mathcal{H}_a) \supseteq \mathfrak{S}_a \ni W_a$ the density operator of the ensemble of measuring apparatuses by which the measurements are performed.
- $\mathcal{B}(\mathcal{H}_o \otimes \mathcal{H}_a) \ni \mathbf{S}$ the unitarian representing the solution of the Schrödinger equation for an interaction of finite duration or a scattering operator.

We remark that unitarity is not an essential requirement. S may contain irreversible dynamics of the measuring apparatus, thus representing a solution of a Schrödinger equation with dissipation. But at least for $\mathcal{B}(\mathcal{H}_o \otimes \mathcal{H}_a) \supseteq \mathfrak{S}_{(o,a)} \ni \mathbf{W}$ the equality

$$\mathrm{tr}\,\mathbf{W} = \mathbf{SWS}^+$$

must hold. Finally:

- $\mathcal{B}(\mathcal{H}_a) \supseteq \{B_j\}_{j\in J}$ is the representing sequence of the operation to be observed on the measuring apparatus.

The measuring process is then described as follows.

We are considering a series of experiments in which the object and the apparatus are prepared independently from one another. The initial state of the combined system is therefore uncorrelated. Let

$$W_o \otimes W_a$$

denote that state. The interaction (which may include irreversible motion) leads to the correlated state

$$\mathbf{S}(W_o \otimes W_a)\mathbf{S}^+$$

On this state an operation is performed. Whether it concerns a macroscopically observable property of the apparatus only or a general property of the combined system as an operation, it must be described by a sequence of operators $\{\mathbf{B}_j\}_{j\in J}$ with $\Sigma_{j\in J}\,\mathbf{B}_j^+\mathbf{B}_j \leq 1$. That we write $1 \otimes B_j$ instead of more general $\mathbf{B}_j$ has only aesthetic reasons and no consequences for the later computations. Let $\mathbf{\Phi}$ denote the complete positive map corresponding to this operation, i.e.,

$$\mathbf{\Phi}(\mathbf{S}(W_o \otimes W_a)\mathbf{S}^+) = \sum_{j\in J} (1 \otimes B_j)\mathbf{S}(W_o \otimes W_a)\mathbf{S}^+(1 \otimes B_j^+)$$

Since we want to describe the operation on the object component of the combined system, which means that we are interested only in the results of further measurements concerning the first component described in the Hilbert space $\mathcal{H}_o$, we can use the partial trace formalism. Let tr_a denote the partial trace with respect to the Hilbert space of the apparatus. With

$$\Phi(W_o) := \mathrm{tr}_a(\mathbf{\Phi}(\mathbf{S}(W_o \otimes W_a)\mathbf{S}^+))$$

the representing map of the desired operation is given. The density operator of the ensemble of objects prepared by selection according to the result '+' on the apparatus is given by

$$\hat{W}_+ = \frac{\Phi(W_o)}{\mathrm{tr}(\Phi(W_o))} \in \mathfrak{S}_o$$

To find the corresponding representing sequence of operators we write the map $\mathbf{\Phi}$ corresponding to the operation on the combined system in the form

$$\mathbf{\Phi}(\mathbf{S}(W_o \otimes W_a)\mathbf{S}^+)$$

$$= \mathrm{tr}_a\left(\sum_{j\in J} (1 \otimes B_j)\mathbf{S}(1 \otimes \sqrt{W_a})(W_o \otimes 1)(1 \otimes \sqrt{W_a})\mathbf{S}^+(1 \otimes B_j^+)\right)$$

We get the map Φ corresponding to the operation on the object system by forming the partial trace tr_a. This becomes more explicit if we write the bilinear form

$$\langle \varphi, \Phi(W_o)\psi\rangle = \langle \varphi, \mathrm{tr}_a\, \mathbf{\Phi}(\mathbf{S}(W_o \otimes W_a)\mathbf{S}^+)\psi\rangle$$

$$= \sum_{j\in J}\sum_{\nu=0}^{\infty} \langle \varphi \otimes \phi_\nu, (1 \otimes B_j)\mathbf{S}(1 \otimes \sqrt{W_a})$$

$$\times (W_o \otimes 1)(1 \otimes \sqrt{W_a})\mathbf{S}^+(1 \otimes B_j^+)\psi \otimes \phi_\nu\rangle$$

where $\{\phi_\nu\}_{\nu\in\mathbf{N}}$ is a complete orthonormal system in the Hilbert space $\mathcal{H}_a$ of the apparatus. This expression suggests we introduce the series of operators $\{A_{j\nu\mu}\}_{j\in J,\nu\in\mathbf{N},\mu\in\mathbf{N}}$ by the definition

$$\langle\varphi, A_{j\nu\mu}\psi\rangle := \langle\varphi\otimes\phi_\nu, (1\otimes B_j)\mathbf{S}(1\otimes\sqrt{W_a})\psi\otimes\phi_\mu\rangle$$

The adjoint operators are then defined by the bilinear forms

$$\langle\varphi, A^+_{j\nu\mu}\psi\rangle := \langle\varphi\otimes\phi_\mu, (1\otimes\sqrt{W_a})\mathbf{S}^+(1\otimes B_j^+)\psi\otimes\phi_\nu\rangle$$

Replacing now the unit operator in the expression $(W_o\otimes 1)$ in the equation for $\Phi(W_o)$ by $\Sigma_{\mu=0}^\infty \mid\phi_\mu\times\phi_\mu\mid$ we get the equality

$$\langle\varphi, \Phi(W_o)\psi\rangle = \sum_{j\in J}\sum_{\nu=0}^\infty\sum_{\mu=0}^\infty \langle\varphi, A_{j\nu\mu}W_oA^+_{j\nu\mu}\psi\rangle$$

Since φ and ψ are arbitrary in $\mathcal{H}$, we have proved

$$\Phi(W_o) = \sum_{j\in J}\sum_{\nu=0}^\infty\sum_{\mu=0}^\infty A_{j\mu\nu}W_oA^+_{j\nu\mu}$$

Hence $\{A_{j\mu\nu}\}_{j\in J,\nu\in\mathbf{N},\mu\in\mathbf{N}}$ is a sequence of operators representing the operation for the object system. Since for $\varphi\in\mathcal{H}_o$, $\|\varphi\| = 1$, we have

$$\begin{aligned}\operatorname{tr}\Phi(|\varphi\rangle\langle\varphi|) &= \operatorname{tr}\sum_{j\in J}\sum_{\nu=0}^\infty\sum_{\mu=0}^\infty A_{j\nu\mu}|\varphi\rangle\langle\varphi|A^+_{j\nu\mu}\\ &= \left\langle\varphi, \sum_{j\in J}\sum_{\nu=0}^\infty\sum_{\mu=0}^\infty A^+_{j\nu\mu}A_{j\nu\mu}|\varphi\right\rangle\end{aligned}$$

and, on the other hand,

$$\operatorname{tr}\Phi(|\varphi\rangle\langle\varphi|) = \mathbf{\Phi}(\mathbf{S}(W_o\otimes W_a)\mathbf{S}^+) \leq 1$$

it follows that

$$F := \sum_{j\in J}\sum_{\nu=0}^\infty\sum_{\mu=0}^\infty A^+_{j\nu\mu}A_{j\nu\mu} \leq 1$$

as it should be.

Considering effects and operations as a result of interaction processes, this form of the representations of operations was derived by K. Kraus and myself at the end of the 1960s by assuming that selections by observations on the apparatus are to be described on a von Neumann–Lüders operation. Later Kraus realized that it is just the Stinespring representation of a complete positive mapping and he gave the more general arguments that this must be fulfilled by the very definition of an operation and the kinematics of coupled

systems. Hence, the artificial assumption about macroscopic observations could be dropped.

The following can be proved: Let be given a complete positive linear mapping Φ operating on $\mathfrak{B}(\mathcal{H}_o)_1$ and a Hilbert space $\mathcal{H}_a$. There are always triplets

$$(W_a, \{B_j\}_{j\in J\subseteq \mathbf{N}}, \mathbf{S}) \in \mathfrak{B}(\mathcal{H}_o)_1 \times (\mathfrak{B}'(\mathcal{H}_o))^2 \times \mathfrak{U}(\mathcal{H}_o \otimes \mathcal{H}_a)$$

where $(\mathfrak{B}'(\mathcal{H}_o))^2$ means that the sum of $B_j^+ B_j$ exists and is a bounded operator, such that Φ arises in the manner just described.

Moreover, one may prove that coexistent effects can be produced together in one and the same interaction process and many other properties fitting well into the philosophy of quantum measurements. The operations described here are the elementary building blocks by which the theory of quantum measurements is formed.

REFERENCES

Busch, P., Lahti, P. J., and Mittelstaed, P. (1991). *The Quantum Theory of Measurements,* Springer, Berlin.

Davies, E. B. (1976). *Quantum Theory of Open Systems,* Academic Press, New York, Chapters 2–4.

Davies, E. B., and Lewis, J. C. T. (1970). An operational approach to quantum probability, *Communications in Mathematical Physics,* **17**, 239–269.

Gudder, S. (1979). *Stochastic Methods in Quantum Mechanics,* Elsevier/North-Holland, Amsterdam, Chapter 4.

Hellwig, K.-E. (1967). Makroskopische Effekte und Quantenmechanische Messung, Dissertation, Universität Marburg.

Hellwig, K.-E. (1969). Coexistent effects in quantum mechanics, *International Journal of Theoretical Physics,* **2**, 147–155.

Hellwig, K.-E. (1971). Measuring process and additive conservation laws, in *Foundation of Quantum Mechanics,* B. d'Espagnat, ed., Academic Press, New York, pp. 338–345.

Hellwig, K.-E., and Kraus, K. (1969). Pure operations and measurements, *Communications in Mathematical Physics,* **11**, 214–220.

Hellwig, K.-E., and Kraus, K. (1970). Operations and measurements II. *Communications in Mathematical Physics,* **16**, 142–147.

Hellwig, K.-E., and Kraus, K. (1971). Formal description of measurements in local quantum field theory, *Physical Review D,* **1**, 566–571.

Kraus, K. (1971). General state changes in quantum theory, *Annals of Physics,* **64**, 311–335.

Kraus, K. (1977). Position observables of the photon, in *The Uncertainty Principle and Foundations of Quantum Mechanics,* W. C. Price and S. S. Chissick, eds., Wiley, London.

Kraus, K. (1983). *States, Effects, and Operations,* Springer, Berlin.

Ludwig, G. (1976). *Einführung in die Grundlagen der Theoretischen Physik,* Vol. 3, Chapter XII, Vieweg, Braunschweig.

Lüders, G. (1951). Über die Zustandsänderung durech den Messprozess, *Annalen der Physik,* **8**, 322–328.

Pauli, W. (1933). Die allgemeinen Prinzipien der Wellemechanik, in *Handbuch der Physik,* Vol. 24, H. Geiger und K. Scheel, eds., Springer, Berlin.